Vol. 12, Issue 05, May: 2024
ISSN(P) 2347-5404 ISSN(O)2320 771X

Different Patterns of Diophantine Equations

FAROOQ AHMAD SHAH
Research Scholar, DAVV University,
Indore M.P

Abstract:
Number theory is an ancient subject and its content is vast, but we still cannot answer the simplest and most natural questions about the integers. Number theory has been called as "The Queen of Mathematics"and Mathematics as "The Queen of Science". The theory of numbers is a branch of mathematics which deals with properties of the whole numbers 1, 2, 3, ... also called the counting numbers or positive numbers.The history of Number Theory can be divided into threeparts, progress of number theory before Christian era, its development in thenext 1500 years and from sixteenth century to present.

The first scientific approach to the study of integers, that is, the true origin of the theory of numbers is generally attributed to the Greeks around 600 BC Pythagoras and his disciples made rather through studies of the integers. The professionalmathematicians are attracted to number theory because of the way of all the weapons of modern mathematics can be brought to bear on its problems.

1. Introduction

Number theory has always fascinated amateurs as well as professional mathematicians. In contrast to other branches of mathematics many of the problems and theorems of number theory can be understood by laypersons, although solutions to the problems and proofs of the theorems often require a sophisticated mathematical background. Modern number theory is a broad subject that is classified its subheadings such as elementary number theory, algebraic number theory, analytic number theory, geometric number theory and probabilistic number theory.

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The understanding of numbers existed in ancient Mesopotamia, Egypt, Chinese and India, for tablets pap and temple carvings from these early cultures have survived, Babylonian tablet known as Plimpton 322(c.1700 BC) is a case in point. In modern notation, it displays number triples x, y and z with the property that $x^2 + y^2 = z^2$. Such triple is 2,291**;** 2,700 and 3,541, where $2{,}291^2 + 2{,}700^2 = 3{,}541^2$. This certainly reveals a degree of number theoretic sophistication in ancient Babylon.

In number theory many problems may be posed as Diophantine equations to be solved in integers. Proving that even simple specific Diophantine equations have no solutions may require very sophisticated methods. There have been instances when the solutions of such problems have emerged only centuries after being posed. And in such cases a lot of deep and beautiful mathematics get generated as result. One such striking example is "The Congruent Number Problems". Congruent number continued to excite the curiosity of number theorists, while ancient times. They fall it the theory of elliptic curves which is a extraordinarily developed theory but still an very important topic of modern research. The ternary third degree equation with three unknowns given by

$3(x2 + y2) - 5xy + x + y + 1 = 15z3$

is studied for its non – zero distinct integral solutions.

In section (3.2), evaluation the solutions of the homogeneous third-degree Diophantine equation

$x3 + y3 = 57(h2 + 3p2)\ zw2$

A few interesting properties among the solutions are presented:

•$x\ (h, p, a, b) + y\ (h, p, a, b) - z\ (h, p, a, b) = 0$

•$[x(h, p, a, b) + y(h, p, a, b)]2 - z2(h, p, a, b) = 0$

•$x\ (1, 1, a, 1) + 7y\ (1, 1, a, 1) + 114\ Proa \equiv 0\ (mod\ 114)$

•$z\ (h, p, a, 1) - h\{6\ Proa - 27Gnoa + 9\} - p\{24\ Proa + 6Gnoa - 66\} = 0$

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The Ternary Third degree Diophantine equation under consideration is

$$3(x^2 + y^2) - 5xy + x + y + 1 = 15z^3 \quad (3.1)$$

Introducing the linear transformations ($u \neq v \neq$ o)

$$x = u + v, \; y = u - v \quad (3.2)$$

In equation (3.1) leads to

$$(u + 1)^2 + 11v^2 = 15z^3 \quad (3.3)$$

Different patterns of solutions of equation (3.1) are presented below:

Write 15 as

$$15 = (2 + i\sqrt{11})\,(2 - i\sqrt{11}) \quad (3.4)$$

Assume that $z = a^2 + 11b^2$ (3.5)

Using equation (3.4) and equation (3.5) in equation (3.3) and applyingthe method of factorization.

$$(u + 1) + i\sqrt{11})\, v = (2 + i\sqrt{11}))\;(a + i\sqrt{11})b)^3 \quad (3.6)$$

Comparing like coefficients on both sides of equation (3.6), we get

$$u = 2a^3 + 121b^3 - 33a^2b - 66ab^2 - 1$$

$$v = a^3 - 22b^3 + 6a^2b - 33ab^2 \quad (3.7)$$

In view of equation (3.7), the solution of equation (3.1) can be written as

$$\left.\begin{aligned} x &= 3a^3 + 99b^3 - 27a^2\,b - 99ab^2 - 1 \\ y &= a^3 + 143b^3 - 39a^2b - 33ab^2 - 1 \\ z &= a^2 + 11b^2 \end{aligned}\right\} \quad (3.8)$$

The equation (3.8) represents non-zero distinct integral solution ofequation (3.1) on two parameters.

Some properties are

- $3z\,(a, a)$ *is a perfect square*
- $2z\,(a, a)$ *is a Nasty number*
- $3[\,y(a, a) + 1]$ *is a cubical integer*

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- $[x(a,-a)+1] is\ a\ perfect\ square$
- $x(a,1)-3y(a,1)-90t_{4,a}+330=0$
- $x(a,1)+30z(a,1)-6P_a^5 \equiv 32(Mod\ 99)$
- $x(a,1)+y(a,1)-8P_a^5+140t_{3,a} \equiv 54(Mod\ 62)$

Equation (3.3) can be written as

$$(u+1)^2+11v^2 = 15z^3 * 1 \tag{3.9}$$

Assume that $z = a^2 + 11b^2$ (3.10)

Write down 1 as

$$1 = \frac{(5+i\sqrt{11}))\ (5-i\sqrt{11}))}{36}$$

$$\text{Also } 15 = (2+i\sqrt{11})\ (2-i\sqrt{11})) \tag{3.11}$$

Using equation (3.10) and equation (3.11) in equation (3.9) and employing the method of factorization.

$(u+1)+i\sqrt{11}v = (2+i\sqrt{11})\ (a+i\sqrt{11}b)^3$ (3.12)

Comparing like coefficients on both sides of equation (3.12), we get

$$\left.\begin{array}{l} u = \frac{1}{6}\{-a^3+847b^3+33ab^2-231a^2b-6\} \\ v = \frac{1}{6}\{7a^3+11b^3-231ab^2-3a^2b\} \end{array}\right\} \tag{3.13}$$

Using equation (3.13) in equation (3.2), the corresponding integersolution of equation (3.1) are given by

$$\left.\begin{array}{l} x = a^3+143b^3-33ab^2-39a^2b-1 \\ y = \frac{1}{6}\{-8a^3+836b^3+264ab^2-228a^2b-6\} \\ z = a^2+11b^2 \end{array}\right\} \tag{3.14}$$

Our interest is to obtain the integer solutions, so that the values of x andy are integers for suitable choices of the parameters a and b.

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Put $a = 6A, b = 6B$

$$\left.\begin{aligned} x &= 216A^3 + 30888B^3 - 7128AB^2 - 8424A^2B - 1 \\ y &= -288A^3 + 30096B^3 + 9504AB^2 - 8208A^2B - 1 \\ z &= 36A^2 + 396B^2 \end{aligned}\right\} \quad (3.15)$$

The equation (3.15) represents non-zero distinct integral solution ofequation (3.1) on two parameters.

Review of Literature

This is the (3.3) equation

$$(u+1)^2 + 11v^2 = 15z^3$$

Again equation (3.3) can be written as

$$(u+1)^2 + 11v^2 = 15z^3 * 1 \quad (3.16)$$

Assume that $z = a^2 + 11b^2$ (3.17)

Write down 15 as

$$\left.\begin{aligned} 15 &= \frac{(6+i3\sqrt{11})(6-i3\sqrt{11})}{9} \\ 1 &= (5+i\sqrt{11})(5-i\sqrt{11}) \div 3 \quad (3.18) \end{aligned}\right\}$$

Using equation (3.18) and equation (3.17) in equation (3.16) andemploying the method of factorization.

$$(u+1) + i\sqrt{11}\, v = 1/18\, \{(6+i3\sqrt{11})(5+i\sqrt{11})(ai\sqrt{11}b)^3\} \quad (3.19)$$

Comparing like coefficients on both sides of equation (3.19), we get

$$\left.\begin{aligned} u &= \tfrac{1}{18}\{-3a^3 + 99ab^2 - 693a^2b + 2541b^3 - 18\} \\ v &= 1/18\{21a^3 - 9a^2b - 693ab^2 + 33b^3\} \end{aligned}\right\} \quad (3.20)$$

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Using equation (3.20) in equation (3.2), the comparable integer solutionof equation (3.1) are given by

$$\left.\begin{aligned} x &= a^3 - 33ab^2 - 39a^2b + 143b^3 - 1 \\ y &= 1/18\ \{-24a^3 + 792ab^2 - 684a^2b + 2508b^3 - 18\} \\ z &= a^2 + 11b^2 \end{aligned}\right\} \quad (3.21)$$

Our interest is to obtain the integer solutions, so that the values of x andy are integers for suitable choices of the parameters a and b.

Replace a by 18A and b by 18B we have

$$\left.\begin{aligned} x &= 5532A^3 - 192456AB^2 - 227448A^2B + 833976B^3 - 1 \\ y &= -7776A^3 + 256608AB^2 - 221616A^2B + 812592B^3 - 1 \\ z &= 324A^2 + 3564B^2 \end{aligned}\right\} \quad (3.22)$$

The equation (3.22) represents non-zero distinct integral solution ofequation (3.1) on two parameters

2. Research Methodology

The present study is based on historical, explanatory and descriptive. The proposed examination has utilized primary and secondary data for examination. It has depended on the narrative investigation of accessible essential source material. The books, investigate articles and other distributed materials comprise the secondary data of the examination.

3. Objectives of the study

1. To study the different Diophantine equations of multi degree polynomial equations.
2. To study for solving homogeneous and non-homogeneous Diophantine equations.
3. To study the importance and need to consider the relation between the integer solutions and special polygonal numbers.

4. Analysis and Results

Again, equation (3.3) can be written as

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$(u+1)^2 + 11v^2 = 15z^3 * 1$ (3.23)

Assume that $z = a^2 + 11b^2$ (3.24)

Write down 15 as

$$\left.\begin{aligned} 15 &= (4 + i2\sqrt{11})(4 - i2\sqrt{11}/4 \\ 1 &= 1 + i3\sqrt{11})(1 - i3\sqrt{11}/100 \end{aligned}\right\} \quad (3.25)$$

Using 3.24 and 3.25 in 3.23,

$$(u+1) + i\sqrt{11}\, v = \frac{1}{20}\{(4 + i2\sqrt{11})(1 + i3\sqrt{11})(a + i\sqrt{11}b)^3\} \quad (3.26)$$

Comparing like coefficients on both sides of equation (3.26), we get

$$u = 1/18\{-62a^3 + 1694b^3 + 2046ab^2 - 462a^2b - 20\} \quad (3.27)$$

$$v = \frac{1}{20}\{682b^3 + 14a^3 - 186a^2b - 462ab^2\} \quad (3.28)$$

Using equation (3.27) and equation (3.28) in equation (3.2), the comparable integer solution of equation (3.1) are given by

$$\left.\begin{aligned} x &= \frac{1}{20}\{-48a^3 + 2376b^3 + 1584ab^2 - 648a^2b - 20\} \\ y &= \frac{1}{20}\{-76a^3 + 1012b^3 + 2508ab^2 - 276a^2b - 20\} \\ z &= a^2 + 11b^2 \end{aligned}\right\} \quad (3.29)$$

The equation (3.30) represents non-zero distinct integral solution ofequation (3.1) on two parameters.

The third degree equation with four unknowns to be solved forobtaining non-zero integral solution is

$x^3 + y^3 = 57(h^2 + 3p^2)zw^2$ (3.31)

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Introducing the linear transformations

$x = u + v,\ \ y = u - v \text{ and } z = 2u$ (3.32)

in equation (3.31) leads to

$u^2 + 3v^2 = 57(h^2 + 3p^2)zw^2$ (3.33)

Now, we solve equation (3.33) through different methods and thusobtain different patterns of solutions to equation (3.31).

Assume that $w = w(a, b) = a^2 + 3b^2$ (3.34)Where

a and b are non-zero distinct integers

Write down 57 as

$$57 = \frac{(3n+4ni\sqrt{3})(3n-4ni\sqrt{3})}{n2} \quad (3.35)$$

Using equation (3.34) and equation (3.35) in equation (3.33) and applying the method of factorization,

$$u + i\sqrt{3}v = (3n + 4ni\sqrt{3})(h + i\sqrt{3}p)(a + i\sqrt{3}b)^2$$

Comparing like terms, we have

$u = h(3a^2 - 24ab - 9b^2) + p(-12a^2 - 18ab + 36b^2)$

$v = h(4a^2 + 6ab - 12b^2) + p(3a^2 - 24ab - 9b^2)$

Hence in view of equation (3.32), the values of x, y, z are given by

$$\left.\begin{aligned} x &= h(7a^2 - 18ab - 21b^2) + p(-9a^2 - 42ab + 21b^2) \\ y &= h(-a^2 - 30ab + 3b^2) + p(-15a^2 + 6ab + 45b^2) \\ z &= h(6a^2 - 48ab - 18b^2) + p(-24a^2 - 36ab + 72b^2) \end{aligned}\right\} \quad (3.36)$$

$w = a^2 + 3b^2$

Thus equation (3.36) represent the non-zero integral solutions toequation (3.31)

Conclusion

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Rewrite equation (3.33) as

$$u^2 + 3v^2 = 57\,(h^2 + 3p^2)\,zw^2 *1 \qquad (3.37)$$

Write down 1 as

$$1 = \frac{(n+ni\sqrt{3})\,(n-ni\sqrt{3})}{(2n)^2} \qquad (3.38)$$

Using equation (3.34), equation (3.35) and equation (3.38) in equation(3.37) and applying the method of factorization.

$$u + i\sqrt{3}v = (n + ni\sqrt{3})\,(3n + 4ni\sqrt{3})(h + i\sqrt{3}\,P)(a + i\sqrt{3}b)^2 \qquad (3.39)$$

Comparing like terms, we have

$$u = \frac{1}{2}\{h\,(-9^2 - 42ab + 27b^2) + p\,(-21a^2 + 54ab + 63b^2)\}$$

$$v = \frac{1}{2}\{h\,(7a^2 - 18ab - 21b^2) + p\,(-9a^2 - 42ab + 27b^2)\}$$

Hence in view of equation (3.32) the values of x, y, z are given by

$$x = \frac{1}{2}\{h\,(-2a^2 - 60ab + 6b^2) + p\,(-30a^2 + 12ab + 90b^2)\}$$

$$y = \frac{1}{2}\{h\,(-16a^2 - 24ab + 48b^2) + p\,(-12a^2 + 96ab + 36b^2)\}$$

$$z = h(-9a^2 - 42ab + 27b^2) + p\,(-21a^2 + 54ab + 63b^2)$$

$$w = a^2 + 3b^2$$

Our interest is to obtain the integer solution, so that the values of x andy are integers for suitable choices of the parameters a and b.

Put $a = 2A, b = 2B$

$$\left.\begin{aligned} x &= h\,(-4A^2 - 120AB + 12B^2) + p\,(-60A^2 + 24AB + 180B^2) \\ y &= h(-32A^2 - 48AB + 96B^2) + p\,(-24A^2 + 192AB + 72B^2) \\ z &= h(-36A^2 - 168AB + 108B^2) + p\,(-84A^2 + 216AB + 252B^2) \end{aligned}\right\} \qquad (3.36)$$

$$w = 4A^2 + 12B^2$$

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Thus equation (3.40) represent the non-zero integral solutions toequation (3.1).

Rewrite equation (3.3) as

$u^2 + 3v^2 = 19 * 3 (h^2 + 3p^2) z w^2/2$ (3.41)

Write down 19 and 3 as

$19 = (4n + ni\sqrt{3})(4n - ni\sqrt{3}/n^2$

$3 = (3n + ni\sqrt{3}(3n - ni\sqrt{3}$

Using equation (3.4) and equation (3.42) in equation (3.41) andapplying the method of factorization.

$u + i\sqrt{3}v = (4n + ni\sqrt{3})\ (3n + ni\sqrt{3})\ (h + i\sqrt{3}P)\ (a + i\sqrt{3}b)^2$ (3.43)

Comparing like terms, we have

$u = \frac{1}{2}\{h(9a^2 - 42ab - 27b^2) + p(-21a^2 - 54ab + 62b^2)\}$

$v = \frac{1}{2}\{h(7a^2 + 18ab - 21b^2) + p(9a^2 - 42ab + 27b^2)\}$

Hence in the view of equation (3.2), the values of x, y, z are given by

$x = 1/2\ \{h(16a^2 - 24ab - 48b^2) + p(-12a^2 - 96ab + 35b^2)$

$y = \frac{1}{2}\{h(2a^2 - 60ab - 6b^2) + p(-30a^2 - 12ab + 89b^2)\}$

$z = h(9a^2 - 42ab - 27b^2) + p(-21a^2 - 54ab + 62b^2)$

$w = a^2 + 3b^2$

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Our interest is to obtain the integer solution, so that the values of x andy are integers for suitable choices of the parameters a and b.

Put $a = 2A, b = 2B$

$$\left.\begin{aligned} x &= h\,(32A^2 - 48AB - 96B^2\,) + p\,(-24A^2 - 192AB + 70B^2) \\ y &= h\,(4A^2 - 120AB - 12B^2\,) + p\,(-60A^2 - 24AB + 178B^2) \\ z &= h\,(36A^2 - 168AB - 108B^2\,) + p\,(-84A^2 - 216AB + 248B^2) \end{aligned}\right\} \quad (3.44)$$

$$w = 4A^2 + 12B^2$$

Thus equation (3.44) represents non-zero integral solutions to equation(3.1).

Reference

1. Gopalan. M.A., Ganapathy R, Srikanth R. "On the Diophantine equation$x^2 + pxy + y^2 = z^2$", Bulletin of Pure and Applied Sciences, Vol LII, No: 1-2, 2000, 15-17.
2. Gopalan. M.A, Anbuselvi.R, "On Ternary Quadratic Homogeneous Diophantine Equation $x^2 + pxy + y^2 = z^2$", Bulletin of Pure and Applied Sciences 2005; 24E(2):405-408
3. Gopalan. M.A, Anbuselvi.R, "Integral Solution of ternary cubic Diophantine Equation $x^2 + y^2 + 4N = zxy$", Bulletin of Pure and Applied Sciences Vol. LXVII,No. 1-2, 107-111, March 2008.
4. Gopalan. M.A, Anbuselvi.R, "Integral Solutions of ternary quadratic equation $x^2 + y^2 = z^4$", Acta Ciencia Indica, Vol XXXIII M, No. 1, 297-300, 2008.
5. Gopalan. M.A, Anbuselvi. R, "A Special Pythagorean Triangle", ActaCiencia Indica, Vol: XXXIM, No:1, PP:053-054, 2005.
6. Gopalan. M.A., Geetha K.,'' On the ternary cubic Diophantine $x^2 + y^2 - xy = z^3$ ", Bessel J. Math., 3(2), 119-123, 2013.
7. Gopalan. M.A, Geetha. V, (2013), Integral Solutions of ternary biquadratic equation $x^2 + 13y^2 = z^4$", IJLRST, Vol.2,Issue 2, 59-61.
8. Gopalan. M.A, Manju Somnath and Vanitha N., "Parametric Solutions of $x^2 - y^6 = z^2$", Acta Ciencia Indica, Vol: XXXIII, No:3, PP:1083-1085, 2007.
9. Gopalan. M.A, Manju Somnath and Vanitha N., "Ternary cubic Diophantine equation $x^2 - y^2 = z^3$", Acta Ciencia Indica, Vol: XXXIIIM, No:3, PP:705-707, 2007.