Vol. 12, Issue 04, April: 2024
ISSN(P) 2347-5404 ISSN(O)2320 771X

A Study of Hypergeometric Functions and their Generalizations

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Abstract:
Special functions are particular mathematical functions which have more or less established names and notations due to their importance in mathematical physics, engineering and other disciplines. We now present a brief sketch of hypergeometric functions and their generalizations since most of the classical special functions can be represented in terms of hypergeometric functions and their generalizations.

The special functions laid their existence in the quest of researchers to find solutions of certain differential equations which occurred as mathematical models of well-known problems in physics. Since each definition was made to meet an exigency, all these functions are termed as special functions. The history of special functions is closely tied to the problems of terrestrial and celestial mechanics that were solved in the eighteenth and beginning of nineteenth centuries. In these cases, the common problem was to solve an ordinary or partial differential equation emerging due to mathematical formulation and modeling. Thus, special functions are mathematical functions catering to special needs in the field of physics, engineering, biology and other inter-disciplinary areas. They form a class of well documented functions with extensive literature. The development of the theory of special functions and its applications went hand in hand. The enormous growth in the volume of research work on special functions witnessed during 18th and 19th century compelled the researches to think in terms of their unification. Their efforts in this direction were rewarding and it was found that a large number of special functions can be put in the form of generalized hypergeometric functions or their limiting cases can be employed to arrive at the corresponding result for these special functions.

In view of this inherent relation between hypergeometric functions and other special functions, the special functions have occupied a place of pride in subsequent research work aimed at unifying the scattered mass of results in the theory of special functions. These functions have also found applications in mathematical statistics. Moreover, generalizations of hypergeometric functions have many interesting fractional integral- inequality. The advent of hypergeometric functions opened many new areas of mathematical approach to research investigation in physics, engineering and recently in biology.

Keywords: *Hypergeometric functions, Generalizations, Study, Analysis, Mathematics*

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1. Introduction

The study of one variable hypergeometric function is more than 200 years old. They appear in the work of Euler, Gauss, Riemann and Kummer. Their integral representations were studied by Barnes and Mellin and several special properties of those functions were derived by Schwarz and Goursat.

The Oxford Professor John Wallis (1616-1703), first used the term “hypergeometric” to denote the series which was beyond ordinary geometric series.
The function

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$$F\left[\begin{matrix}a, \\ b,\end{matrix}\ b, Z\right] = F[a, b\,; c; z] = 2F_1(a, b; c; z) = \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

Where c is neither zero nor a negative integer and

$$(a)_n = a(a+1)(a+2) \ldots \ldots \ldots (a+n-1), n > 1$$

$$(a)_0 = 1, a \neq 0$$

Introduced by the famous German mathematician Gauss, C.F. (1755-1855) in the year 1812, under its convergence conditions is known as hypergeometric function.
Since

$$\lim_{n\to\infty} \left|\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}} Z^{n+1} / \frac{(c)_n n!}{(a)_n (b)_n z^n}\right|$$

$$\lim_{n\to\infty} \left|\frac{(.a+n)(b+n)}{(c+n)(n+1)} Z\right| = |Z|$$

So, long as none of a, b, c, is zero or a negative integer, the series in has the circle |z|<1 as its of convergence. If either or both of a and b are zero or a negative integer, the series terminates and convergence does not enter the discussion. On the boundary$|z| = 1$ of the region of convergence, a sufficient condition for absolute convergence of the series is Re (c-a-b)>0. The hypergeometric series in ia absolutely convergent if Re (c-a-b)>0. Equation is called the hypergeometric series. The special case *a = c* and *b = 1* yields, the elementary geometric series.

$$\sum_{n=0}^{\infty} z^n$$

In the above definition of hypergeometric function, there are only three parameters; we can have any number of parameters in the denominator and numerator leading to the generalization of $2f_1$ function. In fact, the study of hypergeometric functions is of fundamental importance in the field of many theoretical topics of science.

During 18th and 19th century the Gottingen School under Hindenburg (1741-1808) put a great effort on various complicated extensions of the binomial and multinomial theorems. But the credit of first successful extension went to Gauss (1777-1855) who delivered his famous thesis "Disquisitiones generals circa seriem infinitam" before the Royal Society in Gottingen. In it, this brilliant mathematician defined the infinite series,

$$1+\frac{abz}{c\ \ 1!} + \frac{a(a+1)b(b+1)z^2}{c(c+1)2!} + \frac{a\,(a+1)(a+2)b(b+1)(b+2)z^3}{c(c+1)(c+2)\ \ 3!} + \ldots$$

and introduced the notation $F[a, b, c, z]$ .This series is called the *Gauss series*. This series is of great importance to mathematicians. This series is known as the ordinary hyper geometric series and may be regarded as a generalization of the series,

$$1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$$

The quantities$a$, $b$and c are independent of z.This function is usually represented by the symbol $_2F_1[a, b; c; z]$ and$a$,b and c are called the parameters where z is argument of the function. First two parameters are numerator parameters and third parameter is a denominator parameter. All the quantities may be real or complex numbers. The Gauss series can be written in the following manner also.

$$_2F_1[a, b; c; z] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!}$$

Where *(c ≠ 0, -1, -2, . . .)* and the Pochhamer symbol $(a)_n$is as follows by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a\,(a+1)\,(a+2)\ \ldots(a+n-1),\ n \geq 1$$

$$(a)_0 = 1,\ a \neq 0$$

If either of the numerator parameter is a negative integer, the series (above) terminates. The series is not defined if the denominator parameter is non-positive integer.

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ISSN(P) 2347-5404 ISSN(O)2320 771X

If either of the numerator parameters is a negative integer, the series terminates to a polynomial. Gauss gave many relations between two or more forms of these series. He also proved the famous summation theorem, named as Gauss theorem.

$$2F1[a;\ b;\ c;\ 1] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)}.$$

2. Review of literature

The idea of extending the number of parameters in the Gauss function seems to have occurred for the first time in the work of Clausen. He introduced a series with three numerator parameters and two denominator parameters. This idea was further extended to four to five parameters and so on. Over the next hundred years, the well-known set of special summation theorems associated with names of Salschutz, Dixon and Doughall were developed. These are all for the series in which argument is taken as unity. It can be shown that, Doughall's theorem, the summation of $_7F_6$ series, is the most general possible theorem of this kind.

Hypergeometric function $2\ F_1$ has been generalised from time to time by various mathematicians. The generalized hypergeometric function of single variable z has been defined by the series.

$$\mathrm{p}\ F_q\left[\begin{matrix} a_1 \ldots\ldots\ldots a_1 \\ b_1 \ldots\ldots\ldots b_q \end{matrix} |z\right] = \mathrm{p}\ F_q\left[(a_p); (b_q): z\right]$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n}{\prod_{j=1}^{q} (b_j)_n} \frac{z^n}{n!}$$

Where p and q are positive integers or zero (interpreting an empty product as 1) the numerator parameters $a_{1,} \ldots \ldots, a_p$ and the denominator parameters $b_{1,} \ldots \ldots \ldots, b_q$ take on complex values provided that $b_j \neq 0, -1, -2 \ldots \ldots \ldots; j = 1,2, \ldots \ldots, q.$ *the series* converges for all values of z, real or complex, when p ≤ q. if p =q+1, the series converges for |z| <1.

Further if p= q+1, the series is

1.Absolutely convergent for |z|=1, if Re (w)>0, where,

$$w = \sum_{k=1}^{q} b_k - \sum_{k=1}^{p} a_k$$

2.Convergent for z=-1, if -1<Re (w)≤0 and

3.Divergent for |z|= 1, if Re (w)≤-1

If p > q + 1 the series never converges expect when z = 0, the function is only defined when the series terminates. A comprehensive account of hypergeometric function, confluent hypergeometric function and generalized hypergeometric function has been given in the standard works by Eedelyi et al., Exton and Rainville and slater.

Over the researches of several years, the hypergeometric functions were generalized in various ways. In one method the numbers of parameters were increased. Yet another direction towards generalization was to consider contour integral representation. This approach was first considered by Barnes. Thes laid the foundation for the introduction of G, H, and I function.

The G-function was introduced by Cornels Simon Meijer (1936) as a very general function intended to include most of the known special functions as particular cases. This was not the only attempt of its kind. The generalized hypergeometric function and Macrobert E-function had the same aim, but Meijer's G-function was able to include this as particular case as well. The majority of the special functions can be represented in terms of the G-functions.

The first definition was made by Meijer using a series, nowadays the accepted and more general definition is in terms of Mellin-Barnes type integral. Meijer's G-functions provides an interpretation of the symbol pFq when p>q+1.

The Meijer's G-function is defined as

$$G_{p,q}^{mn}\left[z\left|\frac{a_1,a_2,\dots,a_p}{b_1,b_2,\dots,b_q}\right.\right] = G\frac{m,n}{p,q}\left[z\left|\frac{a_p}{b_q}\right.\right] = G_{p,q}^{m,n}(z)$$

$$G_{p,q}^{mn}\left[z\left|\frac{a_1,a_2,\ \dots\ ,a_p}{b_1,b_2,\ \dots\ ,b_q}\right.\right] = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)z^s ds}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}$$

Charles Fox introduced and studied a more general function, known as Fox's H-function. This function contains all the aforementioned functions, including G-function, as its special cases. Fox has defined H-function in terms of a general Mellin-Barnes type integral. He also investigated the most general Fourier kernel associated with the H- function and obtained the asymptotic expansions of the kernel for large values of the argument. Fox has also derived theorems about the H- function as asymmetric Fourier kernel and established certain operational properties for this function.

The H – function is defined by Fox as follows:

$$H(z) = H_{p,q}^{m,n}\left[z/\frac{(a_j,\alpha_j)\ 1,p}{(b_j,\beta_j)1,q}\right] = H_{p,q}^{m,n}\left[z/\frac{(a_j,\alpha_j),\dots,(a_p,\alpha_p)}{(b_j,\beta_j),\dots,(b_q,\beta_q)}\right]$$

$$= \frac{1}{2\pi i}\int_L \varphi(s)z^s ds$$

$$where\varphi(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}$$

Here

(i)$z \neq 0$, z is a complex variable.

(ii)m, n, p and q non-negative integers satisfying 1<n<p and 1<m<q, and for α_j, j = 1, 2, 3, … p and for β_j, j = 1, 2, … q.

(iii)The contour L runs from -i∞ to i∞ such that the poles of
$\Gamma(b_k-\beta_k s)$, k = 1, 2, 3, … , m lies to the right of L and the poles of
$\Gamma(1-a_j+\alpha_j s)$, j = 1, 2, 3, … , n lies to the left of L.

Fox's H-function was never a dead end of generalizations in the field of special functions. The H-function was also generalized into a new type of function in which the denominator parameters are in the summation form of Gamma function products. This was named as the I-function.

The I – function was introduced by Saxena in connection with the solution of a dual integral equations involving sum of H-functions as kernel. It is defined as

$$I(z) = I_{p_i,q_i;r}^{m,n}\left[z/\frac{(a_j,\alpha_j),\dots,(a_p,\alpha_p)}{(b_j,\beta_j),\dots,(b_q,\beta_q)}\right]$$

$$= \frac{1}{2\pi i}\int_L \varphi(s)z^s ds$$

$$\text{Where}\qquad \varphi(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{m}\Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{p_i}(a_{ji}-\alpha_{ji}s)\right\}}$$

3. Research Methodology

1.Collection of literature regarding with the above-entitled research work.
2.Derivation of new results related with study of basic hypergeometric functions with special reference to quantum calculus.
3.Development of techniques/ recent advances for application in different fields.
4.Collection of literature regarding with the above-entitled research work.
5.Derivation of new results related with study of basic hypergeometric functions with special reference to quantum calculus.

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ISSN(P) 2347-5404 ISSN(O)2320 771X

4. Objectives of the study

1. The purpose of the proposed study is to investigate different dimensions of q-fractional calculus and generalizations of basic hypergeometric function which in turn will increase their accessibility to the real-world problems of engineering, science and economics.
2. The proposed study will also focus upon application of fractional and quantum calculus operators to the difference, differential and integral equations arising in real world problems.
3. The study aims to highlight the applications of hypergeometric functions and their generalizations including their basic analogues, in the light of different mathematical disciplines of calculus, like Fractional calculus, q- Fractional calculus, Dirichlet averages and Integral Transforms.
4. The present study also aimed to develop the new techniques for finding the solution of complicated problems in the field of basic hypergeometric functions and their generalizations with special reference to quantum-calculus.

5. Analysis and results

According to exponential representation of a function

$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x}$, he generalized the formula

$\frac{d^m e^{ax}}{dx^m} = a^m e^{ax}$ and

$$\frac{d^v f(x)}{dx^v} = \sum_{n=0}^{\infty} c_n e^{a_n x}\, a_n^v$$

Assumed the derivative of arbitrary order of *f(x)* to be

$$D^v f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x}\, a_n^v$$

This formula is known as Liouville's first definition and has the obvious disadvantage that v must be restricted to the values such that the series converges.

Liouville's**Error! Reference source not found.** second method was applied ton explicit functions of the formx^{-a}, a > 0. He considered the integral

$$I = \int_0^{\infty} u^{a-1} e^{-xu} du$$

The transformation *xu=t* gives the result

$$x^{-a} = \frac{1}{\Gamma a} I$$

Then he obtained by using his first definition operating on both sides

$$D^v x^{-a} = \frac{(-1)^v \Gamma(a+v)}{\Gamma(a)} x^{-a-v}$$

Liouville was successful in applying these definitions to problems in potential theory. The first definition is restricted to certain values of *v* and second is not suitable to the wide class of function.

Under various assumptions (Chebyshev inequality, Grussineguality, minkowski inequality, Hermite-Hamamard inequality, Ostrowski intequality etc.), inequality is playing very significant role in all fields of mathematics, particularly in the theory of approximations. Therefore, in the literature we found several extensions and generalizations of these integral inequalities for the functions of bounded variation, synchronous, lipschitzian, monotonic, absolutely continuous and n- times differentiable mapping etc.

Firstly, we mention below the basic definitions and notations of some well – known operators of fractional calculus, which shall be used in the sequel. Useful and interesting generalization of both the Riemann- Liouville and Erdlyi- kober Fractional integration operators is introduced by Saigo, in terms of gauss's hypergeometric function.

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Let α, β *and* η *are complex numbers* $\epsilon \mathbb{C}$ and let (x $\epsilon \mathbb{R}_+$ the fractional Re(α)>0 and the Fractional derivative (Re(α)<0 of the first kind of a function

$$I^{\alpha,\beta,\eta} \text{ f(x)} = \frac{X^{-\alpha-\beta}}{\Gamma\alpha} \int_0^X (x-t)^{\alpha-1}\; 2F_1[\alpha+\beta, -\eta; 1-\frac{t}{\alpha}]\text{f(t)dt Re }(\alpha)>0$$

q-fractional calculus is the q-extension of the ordinary fractional calculus. The theory of q-calculus operators in the recent past has been applied in the areas like ordinary fractional calculus, optimal control problems, solutions of the q-difference (Differential) and q-integral equations, q-fractional integral inequalities q-transform analysis, and many more. Fractional and -fractional integral inequalities have proved to be one of the most powerful and far-reaching tools for the development of many branches of pure and applied mathematics. These inequalities have gained considerable popularity and importance during the past few decades due to their distinguished applications in numerical quadrature, transform theory, probability, and statistical problems, but the most useful ones are in establishing uniqueness of solutions in fractional boundary value problems and in fractional partial differential equations. A detailed account of such fractional integral inequalities along with their applications can be found in the research contributions by many authors.

6. Conclusion

Here we throws light on the origin and historical developments of special functions like Gauss hypergeometric functions and its Mellin–Barnes integral representation including E- function, Meijer's G-function, Fox's H- function, Saxena'sI-function, Mittag-Leffler function and generalized functions for the fractional calculus (R-function). Generalized hypergeometric functions, basic hypergeometric series (q-series), fractional calculus and its elements have been discussed extensively in the different chapters of the present work. Moreoverthis chapter also gives details of q- fractional calculus and its elements such as q- Reimann-Liouville fractional integral and differential operator, q-weyl operators and q-Saigo's operators etc. have been also discussed extensively in various chapters of this thesis. The purpose of studying theories is to apply them to real world problems. Over the last few years, mathematicians pulled the subject of q-fractional calculus to several applied fields of engineering, science and economics etc. The authors believe that the volume of research in the area of q- fractional calculus will continue to grow in the forth coming years and that it will constitute an important tool in the scientific progress of mankind.

The basic mathematical ideas of fractional calculus (integral and differential operations of non-integer order) were developed long ago by the mathematicians Leibniz (1695), Liouville (1834), Riemann (1892), and others and brought to the attention of the engineering world by Oliver Heaviside in the 1890s, it was not until 1974 that the first book on the topic was published by Oldham and Spanier (1974). Recent monographs and symposia proceedings have highlighted the application of fractional calculus in physics, continuum mechanics, signal processing, and electromagnetic. Here we state some of applications. It may be important to point out that the first application of fractional calculus was made by Abel (1802-1829) in the solution of an integral equation that arises in the formulation of the tautochronous problem.

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