Some Results of Fixed Point Theorem in Dislocated Quasi-Metric Spaces

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Abstract
The purpose of this paper is to the study of fixed point theorems in dislocated quasi-metric spaces and obtains some new results in it and in integral type. Also the paper contains generalized fixed point theorems of F. M. Zeyada [1] et al., C.T. Aage & J. N. Salunke [4] in dislocated quasi-metric space.

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1. Introduction
Let $X$ be a nonempty set and let $d : X \times X \rightarrow [0, \infty)$ be a function satisfying the following conditions:

(i) $d(x, y) = d(y, x) = 0$, implies $x = y$,
(ii) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then $d$ is called a dislocated quasi-metric on $X$. If $d$ satisfies $d(x, x) = 0$, then it is called a quasi-metric on $X$. If $d$ satisfies $d(x, y) = d(y, x)$, then it is called a dislocated metric.

Definition 1.1 Let $X$ be a nonempty set and $p : X \times X \rightarrow [0, \infty)$ be a function. We say $p$ is a partial metric on $X$ if it satisfies the following axioms:

(i) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$,
(ii) $p(x, x) \leq p(x, y)$
(iii) $p(x, y) = p(y, x)$
(iv) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$ for all $x, y, z \in X$

Observe that any partial metric is a dislocated metric. Ultra metric $d$ on $X$ is a metric on $X$ with condition $d(x, y) \leq d(x, z) + d(z, y)$. The studies of partial metric spaces and generalized ultra-metric spaces have application in theoretical computer science [2, 3]. The notion of the dislocated topologies is useful in the context of logic programming. Recently, Zeyada et al [1] have established a fixed point theorem in a complete dislocated
quasi-metric (dq-metric) space, as stated in the following lemma and theorem.

**Lemma 1.1** Let \((X, d)\) be a dq-metric space. If \(f: X \to X\) is a contraction function, then 

\[
\{(f^n(x_0))\} \text{ is a Cauchy sequence for each } x_0 \in X.
\]

**Theorem 1.1** Let \((X, d)\) be a complete dq-metric space and let \(f: X \to X\) be a continuous contraction function. Then \(f\) has a unique fixed point.

**Definition 2.1** A sequence \(\{X_n\}\) in a dq-metric space (dislocated quasi-metric space) \((X, d)\) is called Cauchy if for a given \(\varepsilon > 0\), \(\exists n_0 \in \mathbb{N}\) such that \(\forall m, n \geq n_0\), \(d(X_m, X_n) < \varepsilon\).

In the above definition if we replace \(d(X_m, X_n) < \varepsilon\) or \(d(X_n, X_m) < \varepsilon\) by max \(\{d(X_m, X_n), d(X_n, X_m)\} < \varepsilon\), the sequence \(\{X_n\}\) is called a ‘bi Cauchy.

**Definition 2.2.** A sequence \(\{X_n\}\) dislocated quasi-converges to \(x\) if 

\[
\lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} d(x, x_n) = 0.
\]

In this case \(X\) is called a dq-limit of \(\{X_n\}\).

**Lemma 2.1** Every subsequence of dq-convergent sequence to a point \(x_0\) is dq-convergent to \(x_0\).

**Definition 2.3** A dq-metric space \((X; d)\) is called complete if every Cauchy sequence in it is a dq-convergent.

**Definition 2.4** Let \((X, d_1)\) and \((Y, d_2)\) be dq-metric spaces and let \(f: X \to Y\) be a function. Then \(f\) is continuous if for each sequence \(\{x_n\}\) which is \(d_1\)-convergent to \(x_0\) in \(X\), the sequence \(\{f(x_n)\}\) is \(d_2\)-convergent to \(f(x_0)\) in \(Y\).

**2 Main Results**

**Theorem 3.1** Let \((X, d)\) be a complete dq-metric space and suppose there exist non negative constants \(a_1, a_2, a_3, a_4, a_5\) with \(a_1 + a_2 + 2(a_4 + a_5) < 1\). Let \(f: X \to X\) be a continuous mapping satisfying

\[
d(f(x), f(y)) \leq a_1 d(x, y) + a_2 d(f(x), f(y)) + a_3 d(y, f(y)) + a_4 [d(x, f(x)) + d(y, f(y))] + a_5 [d(x, f(x)) + d(y, f(y))]
\]  

for all \(x, y \in X\). Then \(f\) has a unique fixed point.

Proof: Let \(\{x_n\}\) be a sequence in \(X\), defined as follows. Let \(x_0 \in X\), \(f(x_0) = x_1\), \(f(x_1) = x_2, \ldots, f(x_n) = x_{n+1}, \ldots\) 

\[
d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)) 
\]  

\[
\leq a_1 d(x_{n-1}, x_n) + a_2 d(x_{n-1}, f(x_{n-1})) + a_3 d(x_n, f(x_n)) + a_4 [d(x, f(x)) + d(y, f(y))] + a_5 [d(x, f(x)) + d(y, f(y))]
\]
\[ d(x_{n-1}, x_n) + \alpha_2 d(x_n, x_{n+1}) + \alpha_3 d(x_{n-1}, x_n) + \alpha_4 \left[ d(x_{n-1}, x_n) + d(x_n, x_{n+1}) \right] \leq (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) d(x_{n-1}, x_n) + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) d(x_n, x_{n+1}) \]

Therefore
\[ d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n) \quad \text{where} \quad \lambda = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) / (1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4) \]

Similarly, we have
\[ d(x_{n-1}, x_n) \leq \lambda d(x_{n-2}, x_{n-1}) \]
\[ d(x_n, x_{n+1}) \leq \lambda^2 d(x_0, x_1). \]

Since \( 0 \leq \lambda < 1 \), so for \( n \to \infty \), \( \lambda^n \to \infty \) we have \( d(x_n, x_{n+1}) \to 0 \).

Hence \( \{ x_n \} \) is a Cauchy sequence in the complete dislocated quasi-metric space \( X \), so there is a point \( t_0 \in X \) such that \( x_n \to t_0 \). Since \( f \) is continuous,

\[ f(t_0) = \lim f(x_n) = \lim x_{n+1} = t_0 \]

Thus \( f(t_0) = t_0 \), so \( f \) has a fixed point.

**Uniqueness:** If \( x \in X \) is a fixed point of \( f \), then by (3.1)
\[ d(x, x) = d(fx, fx) \leq [\alpha_1 + \alpha_2 + \alpha_3 + 2(\alpha_4 + \alpha_5)] d(x, x) \]
which is true only if
\[ d(x, x) = 0, \quad 0 \leq \alpha_1 + \alpha_2 + \alpha_3 + 2(\alpha_4 + \alpha_5) < 0 \]
and \( d(x, x) \geq 0 \). Thus \( d(x, x) = 0 \) for a fixed-point \( x \) of \( f \).

Let \( x, y \) be fixed point of \( f' \); then by (3.1)
\[ d(x, y) = d(fx, fy) \leq \alpha_1 d(x, y) + \alpha_2 d(x, x) + \alpha_3 d(y, y) + \alpha_4 [d(x, x) + d(y, y)] + \alpha_5 [d(x, y) + \alpha_1 d(y, x)] \]
i.e. \( d(x, y) \leq (\alpha_1 + 2\alpha_5) d(x, y) \) and from this it follows that \( d(x, y) = 0, \) since \( d(x, y) \geq 0 , \)
\[ 0 \leq (\alpha_1 + 2\alpha_5) < 1. \] Similarly \( d(y, x) = 0. \) Hence \( x = y \), i.e. Uniqueness of the fixed point

**Note:** If \( \alpha_2 = 0 = \alpha_3 \) in (3.1), then \( f \) becomes a contraction map and this shows that

**Theorem 3.2.** Let \((X, d)\) be a complete dq-metric space and let \( f : X \to X \) be a continuous mapping satisfying
\[ d(fx, fy) \leq \alpha \max \{d(x, y), d(x, fx), d(y, fy)\} + \beta \max \{d(x, fy), d(x, y)\} + \mu d(x, y). \]
(3.2)
for all \( x, y \in X \). If \( 0 \leq \alpha, \beta < 1 \) such that \( \alpha + \mu + \beta < 1 \) then \( f \) has a unique fixed point.

**Proof:** Let \( \{x_n\} \) be a sequence in \( X \), defined as follows.

Let \( x_0 \in X \), \( f(x_0) = x_1, f(x_1) = x_2, ..., f(x_n) = x_{n+1}, .... \)
\[ d(x_n, x_{n+1}) = d(fx_{n-1}, fx_n) \leq \alpha \max \{d(x_{n-1}, x_n), d(x_{n-1}, fx_{n-1}), d(x_n, fx_n)\} + \beta \max \{d((x_{n-1}, fx_n), d(x_n, x_n)\} + \mu d(x_{n-1}, x_n) \]
\[ = \alpha \max \{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\} + \beta \max \{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \]
d(x_{n-1},x_n)+\mu d(x_n,x_n)

Case-1

When \( \max \{ d(x_{n-1},x_{n+1}), d(x_{n-1},x_n), d(x_{n-1},x_{n+1}) \} = d(x_{n-1},x_n) \)
\[
d(x_{n-1},x_n) \leq \gamma d(x_{n-1},x_n) + \mu d(x_{n},x_{n+1})
\]

Thus,
\[
d(x_{n},x_{n+1}) \leq \gamma d((x_{1},x_{0})
\]

Since \( 0 \leq \gamma < 1 \), as \( n \to \infty \), \( \gamma^n \to \infty \). Hence \( \{ x_n \} \) is a dq-Cauchy sequence in \( X \). Thus \( \{ x_n \} \) dislocated quasi-converges to some \( t_0 \). Since \( f \) is continuous, we have \( f(t_0) = \lim f(x_n) = \lim x_{n+1} = t_0 \)

Thus \( f(t_0) = t_0 \) that is \( f \) has a fixed point \( t_0 \)

Case-2

When \( \max \{ d(x_{n-1},x_{n+1}), d(x_{n-1},x_n), d(x_{n-1},x_{n+1}) \} = d(x_{n-1},x_{n+1}) \)
\[
\leq d(x_{n-1},x_n) + d(x_{n},x_{n+1})
\]
Therefore
\[
d(x_{n},x_{n+1}) \leq \beta \{ d(x_{n-1},x_n) + d(x_{n},x_{n+1}) \}
\]
\[
(1-\beta) d(x_{n},x_{n+1}) \leq \beta d(x_{n-1},x_n)
\]
\[
d(x_{n},x_{n+1}) \leq \beta / (1-\beta) d(x_{n-1},x_n)
\]
\[
d(x_{n},x_{n+1}) \leq \delta d(x_{n-1},x_n) \quad \text{Where} \quad \delta = \beta / (1-\beta) < 1
\]

Uniqueness: Let \( x \) be a fixed point of \( f \), then by (3.2) \( d(x; x) = d(fx; fx) \leq \gamma \max d(x; x) \)

Where \( \gamma = \alpha + \mu + \beta \)
i.e. \( d(x; x) \leq \gamma d(x; x) \), which gives \( d(x; x) = 0 \), since \( 0 \leq \gamma < 1 \) and \( d(x; x) \geq 0 \).
Thus \( d(x; x) = 0 \) if \( x \) is a fixed point of \( f \).
Let, \( x; y \in X \) be fixed points of \( f \). That is, \( fx = x; fy = y \). Then by (3.2),
\[
d(x; y) = d(fx; fy) \leq \alpha \max \{ d(x; y), d(x; x), d(y; y) \} + \beta \max \{ d(x; y), d(x; y) \} + \mu d(x; y)
\]
\[
= (\alpha + \mu + \beta) d(x; y) \quad \text{which is true only if} \quad d(x; y) = 0 \quad \text{since} \quad d(x; x) = 0 = d(y; y); 0 \leq \gamma < 1.
\]
Similarly \( d(y; x) = 0 \) and hence \( x = y \). Thus \( x \) is a fixed point of \( f \) is unique.

3. Conclusion

The conclusion of this paper is to the study of fixed point theorems in dislocated quasi-metric spaces. These metrics play a very important role not only in topology but also in other branches of science involving mathematics especially in logic programming and electronic engineering and also prove fixed point theorem using rational type of contractive condition.
which generalized the Banach contraction principle in complete metric space and obtain some new results in integral type.

References